

Scalar Representation and Conjugation of Set-Valued Functions

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Abstract

We are considering functions with values in the power set of a pre-ordered, separated locally convex space with closed convex images. To each such function, a family of scalarizations is given which completely characterizes the original function. A concept of a Legendre–Fenchel conjugate for set-valued functions is introduced and identified with the conjugates of the scalarizations. To the set-valued conjugate, a full calculus is provided, including a biconjugation theorem, a chain rule and weak and strong duality results of Fenchel–Rockafellar type.

Keywords: Set-valued function; Legendre–Fenchel conjugate; biconjugation theorem; Fenchel–Rockafellar duality

Classcode: 49N15, 54C60, 90C46

1 Introduction

In this paper, we will introduce a new duality theory for set-valued functions. We will apply our theory proving a biconjugation theorem, a sum- and a chain-rule and weak and strong duality results of Fenchel–Rockafellar type. Apart from the purely academic interest, investigating set-valued functions is motivated by their applicability in financial mathematics, compare [5, 18, 23], and in vector optimization.

The common lack of infima and suprema in vector spaces makes it obvious that ‘vectorial’ constructions are in general not appropriate to solve minimization problems on a vector space, compare [15]. Even if investigations are restricted to order complete, partially ordered spaces or subsets, least upper or greatest lower bounds can be ‘far away’ and have little in common with the function in question.

An approach to solve this dilemma is to search for a set of minimal elements and hereby transform the original vector-valued problem into a set-valued problem.

Apparently, as pointed out by J.Jahn, ‘the best soccer team is not necessarily the team with the best player’, thus we understand an optimal solution of a set-valued minimization problem to be a set rather than a single point. Consequently, we investigate set-valued functions understanding the images to be elements of the power set of a vector space.

The basic idea is to extend the order on the space Z , given by a convex cone $C \subseteq Z$ to an order on $\mathcal{P}(Z)$ in an optimal way, compare [14, 17, 22] and identify a function $f : X \rightarrow \mathcal{P}(Z)$ with its epigraphical extension $f_C(x) = f(x) + C$. This extension can of course be done for vector-valued functions, thus as a special amenity our theory includes vector-valued

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functions as a special class of set-valued functions. We identify the subset $\mathcal{P}^\Delta \subseteq \mathcal{P}(Z)$ as the set of all elements $A = A + C \in \mathcal{P}(Z)$. This set is an order complete lattice w.r.t. \supseteq and contains all images of functions of epigraphical type. Equipped with an appropriate addition and multiplication with nonnegative reals, \mathcal{P}^Δ is a so-called inf-residuated conlinear space, compare [17, 19], thus supplies sufficient algebraic structure to introduce coherent definitions of conjugates, directional derivatives and subdifferentials. At present, we will concentrate on the investigation of conjugates, more can be found in [29], compare also [8] and especially [26] for closely related approaches.

The introduced notion of conjugate functions turns out to be an example of a $(*, s)$ -duality, see [12], and many properties from the well-known scalar conjugate can be rediscovered in our setting.

In [15, 16], duality results are proven directly, using separation arguments in the space $X \times Z$. In contrast, we introduce a family of scalar functions related to and completely characterizing a set-valued function. The set-valued conjugate is also determined by the family of conjugates of the scalarizations of the original function.

Because of the one-to-one correspondence between set-valued functions and their conjugates on the one hand, the family of scalarizations and their conjugates on the other, we can utilize results from the scalar theory to obtain results for the set-valued case.

The article is organized as follows. In Section 2 we resume basic definitions and facts about conlinear spaces, residuation, extended real-valued and set-valued functions. Apart from an extended chain-rule, everything presented in this section can be found in the literature, compare e.g. [11, 17, 19] and also [21, 33]. In Section 3, a family of scalar functions associated to a set-valued function is identified. This family of scalarizations proves to characterize its assigned set-valued counterpart and will prove useful in later course to provide a concise approach to a theory of convex analysis for set-valued functions. The definitions of conjugate and biconjugate are provided in Section 4, while a selection of duality results is presented in Section 5.

2 Preliminaries; Residuation and Conlinear Structure

2.1 General conlinear spaces

In [17, Section 2.1.2], the concept of a conlinear space has been introduced. A set $Z = (Z, +, \cdot)$ is called a conlinear space, if $(Z, +)$ is a commutative monoid with neutral element and for all $z, z_1, z_2 \in Z$, $r, s \in \mathbb{R}_+$ holds $r(z_1 + z_2) = rz_1 + rz_2$, $r(sz) = (rs)z$ and $1z = z$, $0z = 0$.

If (Z, \leq) is a complete lattice and \leq is compatible with the algebraic structure in $(Z, +, \cdot)$, then $(Z, +, \cdot, \leq)$ is called an order complete conlinear space.

If additionally the operation $+: Z \times Z \rightarrow Z$ satisfies $(x + \inf M) = \inf_{m \in M} (x + m)$, then the inf-residuation or inf-difference $\dashv: Z \times Z \rightarrow Z$ defined by

$$\forall z_1, z_2 \in Z : \quad z_1 \dashv z_2 := \inf \{t \in Z \mid z_1 \leq z_2 + t\} \quad (2.1)$$

replaces the usual difference operation. It holds $z_1 \leq z_2 + (z_1 \dashv z_2)$ and $Z = (Z, +, \cdot, \leq)$ is called an inf-residuated order complete conlinear space.

Dually, if $(x + \sup M) = \sup_{m \in M} (x + m)$, then the sup-residuation (sup-difference) $\bar{\dashv}: Z \times Z \rightarrow Z$ defined by

$Z \times Z \rightarrow Z$ defined by

$$\forall z_1, z_2 \in Z : \quad z_1 \dot{-} z_2 = \sup \{t \in Z \mid z_2 + t \leq z_1\}$$

replaces the difference operation. It holds $z_2 + (z_1 \dot{-} z_2) \leq z_1$ and $Z = (Z, +, \cdot, \leq)$ is called a sup-residuated order complete conlinear space.

A conlinear inf-residuated order complete space as image space supplies sufficient structure to apply the concepts introduced in [12] and to define convexity of functions, as multiplication with positive reals is defined as was done in [15]. Also, the structure carries over from a space Z to the set of all subsets, $\mathcal{P}(Z)$, provided the algebraic and order relations are extended to relations on $\mathcal{P}(Z)$ in an appropriate way, see [17, Theorem 13] and [19] for more details.

Residuation seems to be introduced by Dedekind, [7, p. 71], [6, p. 329-330], compare also [3, XIV, §5], [10, chap. XII] or [11, Chap. 3] on sup-residuation. In [11, Lemma 3.3] the concept of inf-residuation is indicated. The corresponding residuations may substitute the difference operation on \mathbb{R} .

2.2 The extended real numbers

The set \mathbb{R} is extended to $\overline{\mathbb{R}}$ in the usual way by adding two elements $+\infty$ and $-\infty$. Setting

$$\begin{aligned} \forall t \in \mathbb{R} : \quad & -\infty \leq t \leq +\infty; \\ \inf \emptyset &= +\infty, \quad \sup \emptyset = -\infty, \end{aligned}$$

the set $(\overline{\mathbb{R}}, \leq)$ is an order complete lattice. Multiplication with nonnegative reals is extended to $\overline{\mathbb{R}}$ by setting $t \cdot (\pm\infty) = \pm\infty$, if $t > 0$ and $0 \cdot (\pm\infty) = 0$. The addition $+$ on \mathbb{R} admits two distinct extensions to an operator on $\overline{\mathbb{R}}$, the inf- and sup-addition.

$$\forall r, s \in \overline{\mathbb{R}} : \quad r \dot{+} s = \inf \{a + b \mid a, b \in \mathbb{R}, r \leq a, s \leq b\}; \quad (2.2)$$

$$r \dot{+}_+ s = \sup \{a + b \mid a, b \in \mathbb{R}, a \leq r, b \leq s\}. \quad (2.3)$$

These constructions have been introduced in [6, p. 329-330] in order to extend the addition from the set of rational to the set of real numbers.

Both inf- and sup-addition are commutative and compatible with the usual order \leq on $\overline{\mathbb{R}}$. The greatest element $+\infty$ dominates the inf-addition $\dot{+}$, $-\infty$ dominates the sup-addition,

$$\forall r \in \overline{\mathbb{R}} : \quad (+\infty) \dot{+} r = +\infty, \quad (-\infty) \dot{+}_+ r = -\infty. \quad (2.4)$$

The sets

$$\mathbb{R}^\Delta = (\overline{\mathbb{R}}, \dot{+}, \cdot, \leq), \quad \mathbb{R}^\nabla = (\overline{\mathbb{R}}, \dot{+}_+, \cdot, \leq)$$

are inf- and sup-residuated complete conlinear spaces, respectively. The suffix Δ indicates inf-residuation and dually the suffix ∇ indicates sup-residuation. Multiplication with -1 is defined as $(-1) \cdot (\pm\infty) = \mp\infty$. We abbreviate $(-1)r = -r$, when no confusion can arise. Obviously, for $M \subseteq \overline{\mathbb{R}}$ and $r, s \in \overline{\mathbb{R}}$ the following is satisfied.

$$\begin{aligned} (-1) \inf M &= \sup(-1)M; \\ r \dot{+}(-s) &= r \dot{-} s, \quad -(r \dot{+} s) = (-s) \dot{+}_+ (-r); \\ r \dot{+}_+(-s) &= r \dot{-}_+ s, \quad -(r \dot{+}_+ s) = (-s) \dot{+}(-r). \end{aligned}$$

Multiplication with -1 is a duality between \mathbb{R}^Δ and \mathbb{R}^∇ .

2.3 Extended real-valued functions

Let X and Y be locally convex separated spaces with topological duals X^* and Y^* and $g : X \rightarrow \mathbb{R}^\Delta$ a function. Multiplication with -1 transfers g to $-g : X \rightarrow \mathbb{R}^\nabla$. We will concentrate on the first type of functions in the sequel, keeping in mind that for the second class symmetric results can be proven. To $g : X \rightarrow \mathbb{R}^\Delta$, the epigraph and effective domain of g are defined as usual

$$\text{epi } g = \{(x, r) \in X \times \mathbb{R} \mid g(x) \leq r\}, \quad \text{dom } g = \{x \in X \mid g(x) \neq +\infty\}.$$

The set $(\mathbb{R}^\Delta)^X = \{g : X \rightarrow \mathbb{R}^\Delta\}$ equipped with the point-wise addition, multiplication with nonnegative reals and order relation is an order complete, inf-residuated conlinear space. The inf-convolution with respect to either inf-addition or sup-addition is denoted as \square^\cdot and \square_\cdot , respectively. We denote

$$\forall x \in X : \quad (fT)(x) = f(Tx); \quad (Tg)(y) = \inf_{Tx=y} g(x) \quad (2.5)$$

for a linear continuous operator $T : X \rightarrow Y$, $f : Y \rightarrow \mathbb{R}^\Delta$ and $g : X \rightarrow \mathbb{R}^\Delta$.

A function $g : X \rightarrow \mathbb{R}^\Delta$ is said to be closed (convex), if its epigraph is a closed (convex) set. It is subadditive, if $\text{epi } g$ is closed under addition, i.e. $\text{epi } g + \text{epi } g \subseteq \text{epi } g$ and positively homogeneous, if $\text{epi } g$ is a cone, i.e.

$$\text{epi } g = \text{cone}(\text{epi } g) = \{t(x, r) \in X \times \mathbb{R} \mid t > 0, (x, r) \in \text{epi } g\}.$$

A positively homogeneous convex function is called sublinear. Equivalently, a function is sublinear if and only if it is positively homogeneous and subadditive.

A function $g : X \rightarrow \mathbb{R}^\Delta$ is proper if $\text{dom } g \neq \emptyset$ and for all $x \in X$ exists a $t \in \mathbb{R}$ such that $(x, t) \notin \text{epi } g$.

A well known separation theorem is the following corollary of the Hahn Banach Theorem.

Lemma 2.1 [9, Corollary 1.4] *In a locally convex separated space Z , every closed convex set is the intersection of all closed (affine) half-spaces containing it.*

As each such affine half-space can be expressed via a non-zero affine continuous function, $H_\alpha(z^*) = \{z \in Z \mid \alpha \leq -z^*(z)\}$, it is immediate that a function $g : X \rightarrow \mathbb{R}$ is closed, convex and either proper or constant $+\infty$ or $-\infty$, if and only if it is the supremum of its affine minorants, in which case the epigraph of g as a subset of $X \times \mathbb{R}$ is the intersection of all $H_\alpha(x^*, -1)$, where $x^*(x) + \alpha \leq g(x)$ for all $x \in X$. A similar result will be provided for set-valued functions in Theorem 3.13. Equivalently to Lemma 2.1, one can state that in a locally convex separated space Z , whenever there is a closed convex set $M \subseteq Z$ and an element $z \notin M$, then it exists a non-zero element $z^* \in Z^* \setminus \{0\}$ and $t \in \mathbb{R}$, such that

$$-z^*(z) < t \leq \inf_{m \in M} -z^*(m). \quad (2.6)$$

Equation (2.6) will prove useful in various occasions throughout this paper.

The conjugate and biconjugate of $g : X \rightarrow \mathbb{R}^\Delta$ are given by

$$\forall x^* \in X^* : \quad g^*(x^*) = \sup_{x \in X} (x^*(x) \dot{-} g(x)); \quad (2.7)$$

$$\forall x \in X : \quad g^{**}(x) = \sup_{x^* \in X^*} (x^*(x) \dot{-} g^*(x^*)). \quad (2.8)$$

For completeness we cite the fundamental duality formula, Theorem 2.2 and sum up these considerations with an extended chain-rule for scalar valued functions. To our ends, it is of no consequence that in [33] the product of $0 \cdot (+\infty)$ is defined differently from our definition.

Theorem 2.2 [33, Theorem 2.7.1(iii)] *Let X and Y be topological linear spaces with topological duals X^* and Y^* and $h : X \times Y \rightarrow \overline{\mathbb{R}}$ a proper convex function with $(x_0, 0) \in \text{dom } h$ for some $x_0 \in X$. If $h(x_0, \cdot) : Y \rightarrow \overline{\mathbb{R}}$ is continuous in $0 \in Y$, then*

$$\inf_{x \in X} h(x, 0) = \sup_{y^* \in Y^*} 0 \dot{-} h^*(0, y^*);$$

$$\exists y^* \in Y^* : \quad \inf_{x \in X} h(x, 0) = 0 \dot{-} h^*(0, y^*).$$

Moreover, if $\bar{x} \in X$, then

$$\exists \bar{y}^* \in Y^* : \quad (0, \bar{y}^*)(\bar{x}, 0) \dot{-} h(\bar{x}, 0) = h^*(0, \bar{y}^*)$$

$$\Leftrightarrow h(\bar{x}, 0) = \inf_{x \in X} h(x, 0).$$

Theorem 2.3 (Scalar Chain Rule) *Let Y be another topological linear space with topological dual Y^* , $g : X \rightarrow \mathbb{R}^\Delta$, $f : Y \rightarrow \mathbb{R}^\Delta$ two functions and $T : X \rightarrow Y$, $S : Y \rightarrow X$ linear continuous operators.*

(a) *The conjugate of $(g \square \cdot Sf)$ is the function*

$$\forall x^* \in X^* : \quad (g \square \cdot Sf)^*(x^*) = (g^* \dot{+} f^* S^*)(x^*) \leq (g^* \dot{+} f^* S^*)(x^*).$$

(b) *The conjugate of $(g \dot{+} fT)$ is dominated by $(g^* \square \cdot T^* f^*) : X^* \rightarrow \mathbb{R}^\Delta$,*

$$\forall x^* \in X^* : \quad (g \dot{+} fT)^*(x^*) \leq (g^* \square \cdot T^* f^*)(x^*) \leq (g^* \square \cdot T^* f^*)(x^*).$$

(c) *If additionally g or f is the constant mapping $+\infty$, then*

$$\forall x^* \in X^* : \quad -\infty = (g \dot{+} fT)^*(x^*) = (g^* \square \cdot T^* f^*)(x^*);$$

$$\forall x^* \in X^*, \forall y^* \in Y^* : \quad (g \dot{+} fT)^*(x^*) = g^*(x^* - T^* y^*) \dot{+} f^*(y^*).$$

(d) *If $(fT)(x_0) = -\infty$ is satisfied for some $x_0 \in \text{dom } g$ or if both f and g are proper, convex and f is continuous in a point in $T(\text{dom } g)$, then*

$$\forall x^* \in X^* : \quad -\infty \neq (g \dot{+} fT)^*(x^*) = (g^* \square \cdot T^* f^*)(x^*);$$

$$\exists y^* \in Y^* : \quad (g \dot{+} fT)^*(x^*) = g^*(x^* - T^* y^*) \dot{+} f^*(y^*).$$

PROOF.

(a) We apply calculus rules in the residuated spaces \mathbb{R}^Δ and \mathbb{R}^∇ to obtain the following for all $x^* \in X^*$.

$$\begin{aligned} (g \square \cdot Sf)^*(x^*) &= \sup_{x \in X} \left(x^*(x) \dot{-} \inf_{\bar{x} \in X} \left(g(x - \bar{x}) \dot{+} \inf_{Sy = \bar{x}} f(y) \right) \right) \\ &= \sup_{x \in X, y \in Y} \left((x^*(x - Sy) \dot{-} g(x - Sy)) \dot{+} (S^* x^*(y) \dot{-} f(y)) \right) \\ &= g^*(x^*) \dot{+} (f^* S^*)(x^*) \end{aligned}$$

and the second inequality holds by definition of $\dot{+}$ and $\dot{-}$.

(b) First we consider $T^*y^* = x^*$:

$$(fT)^*(x^*) = \sup_{x \in X} (y^*(Tx) \dot{-} f(T(x))) \leq \sup_{y \in Y} (y^*(y) \dot{-} f(y)) = f^*(y^*).$$

We apply results from residuation theory to prove the following for all $x^*, \bar{x}^* \in X^*$.

$$\begin{aligned} (g \dot{+} fT)^*(x^*) &= \sup_{x \in X} ((x^*(x) + \bar{x}^*(x) - \bar{x}^*(x)) \dot{+} (-1)(g(x) \dot{+} (fT)(x))) \\ &= \sup_{x \in X} (((x^* - \bar{x}^*)(x) \dot{-} g(x)) \dot{+} (\bar{x}^*(x) \dot{-} (fT)(x))) \\ &\leq g^*(x^* - \bar{x}^*) \dot{+} (fT)^*(\bar{x}^*) \\ &\leq g^*(x^* - \bar{x}^*) \dot{+} T^*f^*(\bar{x}^*) \end{aligned}$$

and the second inequality holds by definition of $\dot{+}$ and $\dot{-}$.

(c) and the improper case of (d) are proven by direct calculation, while the proper case is classic, compare e.g. [21, Chap. 3, §3.4 Theorem 1]. \square

Example 2.4 Under the assumptions of Theorem 2.3, let $g \equiv +\infty$ and $f(Tx_0) = -\infty$ with $x_0 \in X$. Then

$$\begin{aligned} \forall x \in X : \quad (g \square Sf)(x) &= (g \dot{+} fT)(x) = +\infty; \\ \forall x^* \in X^* : \quad g^*(x^*) &= -\infty; \quad f^*S^*(x^*) = T^*f^*(x^*) = +\infty. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &= (g \square Sf)^*(x^*) \leq (g^* \dot{+} f^*S^*)(x^*) = +\infty; \\ -\infty &= (g \dot{+} fT)^*(x^*) \leq (g^* \square T^*f^*)(x^*) = +\infty \end{aligned}$$

for all $x^* \in X^*$. The statements of Theorem 2.3(a) and (c) do not apply with equality for the inf-addition on the right hand side. In general, sup-addition is dominated by inf-addition and both operators coincide if neither addend is $-\infty$.

2.4 The set \mathcal{P}^Δ

In the sequel, we will consider X , Y and Z to be locally convex separated spaces with topological duals X^* , Y^* , Z^* and investigate on functions from X to a subset \mathcal{P}^Δ of the power-set $\mathcal{P}(Z)$ which is an order complete, inf-residuated conlinear space. The suffix Δ indicates inf-residuation and dually the suffix ∇ indicates sup-residuation.

On Z , a reflexive and transitive order is given by a convex cone $C \subseteq Z$ with $\{0\} \subsetneq C$, setting $z_1 \leq z_2$, iff $z_2 - z_1 \in C$. The negative dual cone of C is C^- , defined by

$$C^- = \{z^* \in Z^* \mid \forall c \in C : z^*(c) \leq 0\}$$

and we assume $\{0\} \subsetneq C^-$. To an element $z^* \in Z^*$ we define $H(z^*) = \{z \in Z \mid 0 \leq -z^*(z)\}$. Obviously, $C \subseteq H(z^*)$, if $z^* \in C^-$ and $\text{cl } C = \bigcap_{z^* \in C^-} H(z^*)$. We do not assume the topological interior of C to be nonempty.

On $\mathcal{P}(Z)$, we introduce an algebraic structure by defining

$$\begin{aligned} \forall A, B \in \mathcal{P}(Z) : \quad A + B &= \{a + b \mid a \in A, b \in B\}, \\ \forall t \in \mathbb{R} \setminus \{0\} : \quad tA &= \{ta \mid a \in A\}, \end{aligned}$$

the Minkowsky sum of two sets and the product of a set with a real number $t \neq 0$. By convention, $A + \emptyset = \emptyset + A = \emptyset$ and $t\emptyset = \emptyset$ for all $A \in \mathcal{P}(Z)$ and $t \neq 0$ while $0A = \{0\}$ for all $A \in \mathcal{P}(Z)$. We abbreviate $z + A = \{z\} + A$ and $-A = (-1)A$ as well as $A - B = A + (-1)B$ for $A, B \in \mathcal{P}(Z)$ and $z \in Z$.

The order relation in Z can be extended in two distinct ways by setting

$$\begin{aligned} \forall A, B \in \mathcal{P}(Z) : \quad A \preceq_C B &\Leftrightarrow B \subseteq A + C; \\ \forall A, B \in \mathcal{P}(Z) : \quad A \preceq_C B &\Leftrightarrow A \subseteq B - C, \end{aligned}$$

see [14, 17, 25].

Two sets $A, B \in \mathcal{P}(Z)$ are equivalent with respect to \preceq_C , iff $A + C = B + C$. We identify the set $\mathcal{P}^\Delta(Z, C) = \{A \in \mathcal{P}(Z) \mid A = A + C\}$ and abbreviate $\mathcal{P}^\Delta = \mathcal{P}^\Delta(Z, C)$, if no confusion can arise. In \mathcal{P}^Δ , $A \preceq_C B$ is equivalent to $B \subseteq A$. The set \mathcal{P}^Δ is a complete lattice, infimum and supremum of a nonempty set $\mathcal{M} \subseteq \mathcal{P}^\Delta$ are given by

$$\inf \mathcal{M} = \bigcup_{M \in \mathcal{M}} M \in \mathcal{P}^\Delta; \quad \sup \mathcal{M} = \bigcap_{M \in \mathcal{M}} M \in \mathcal{P}^\Delta$$

and by convention $\inf \emptyset = \emptyset$ and $\sup \emptyset = Z$. The greatest element in \mathcal{P}^Δ is \emptyset , the smallest Z . For any $\mathcal{M} \subseteq \mathcal{P}^\Delta$ and $A \in \mathcal{P}^\Delta$,

$$\inf(A + \mathcal{M}) = A + \inf \mathcal{M}; \quad \sup(A + \mathcal{M}) \supseteq A + \sup \mathcal{M}$$

is satisfied. Altering the multiplication with 0 to $0A = C$ for all $A \in \mathcal{P}^\Delta$, the set \mathcal{P}^Δ together with the Minkowsky sum, the altered multiplication with nonnegative reals and the order relation \supseteq is an order complete, inf-residuated conlinear space. According to Equation (2.1), the corresponding inf-residuation is given by

$$A \dot{-} B = \inf \{M \in \mathcal{P}^\Delta \mid A \preceq_C B + M\} = \{z \in Z \mid A \supseteq B + z\},$$

In case the ordering cone is set to $C = \{0\}$, this operation is sometimes called the star difference, compare [28, Section 4] and the references therein.

Lemma 2.5 *For $z^* \in Z^*$, $\alpha \in \overline{\mathbb{R}}$ and $A \subseteq Z$, set $\sigma(z^*|A) = \sup \{z^*(a) \mid a \in A\}$, $\sigma(z^*|\emptyset) = -\infty$ and*

$$H_\alpha(z^*) = \{z \in Z \mid \alpha \leq -z^*(z)\}.$$

Then $H(z^) = H_0(z^*)$, $H_{\alpha+(-\sigma(z^*|A))}(z^*) = \text{cl}(A + H_\alpha(z^*))$ and $H_\alpha(z^*) \dot{-} A = H_{\alpha \dot{-} (-\sigma(z^*|A))}(z^*)$.*

PROOF. It holds

$$\text{cl}(A + H_\alpha(z^*)) = \text{cl} \bigcup_{a \in A} (a + H_\alpha(z^*))$$

and either $A = \emptyset$ and $H_{\alpha \dot{+} (-\sigma(z^*|A))}(z^*) = \text{cl}(A + H_\alpha(z^*)) = \emptyset$, or $(a + H_\alpha(z^*)) = H_{\alpha \dot{+} (-\sigma(z^*(a)))}(z^*)$ is satisfied for all $a \in A$. Moreover,

$$\text{cl} \bigcup_{a \in A} H_{\alpha \dot{+} (-\sigma(z^*(a)))}(z^*) = H_{\inf_{a \in A} (\alpha \dot{+} (-\sigma(z^*(a))))}(z^*)$$

is true and by the calculus rules in \mathbb{R}^Δ ,

$$\inf_{a \in A} (\alpha \dot{+} (-\sigma(z^*(a)))) = \alpha \dot{+} (-\sigma(z^*|A)),$$

hence $H_{\alpha \dot{+} (-\sigma(z^*|A))}(z^*) = \text{cl}(A + H_\alpha(z^*))$. Likewise, by applying the residuation rules one proves

$$H_\alpha(z^*) \dot{-} A = H_{\alpha \dot{-} (-\sigma(z^*|A))}(z^*).$$

□

We abbreviate

$$\mathcal{P}^\Delta = (\mathcal{P}^\Delta, +, \cdot, \supseteq); \quad \mathcal{P}^\nabla = (\{A - C \mid A \in \mathcal{P}(Z)\}, +, \cdot, \subseteq). \quad (2.9)$$

Multiplication with -1 is a duality between \mathcal{P}^Δ and the order complete, sup-residuated conlinear space \mathcal{P}^∇ . With convex duality in mind, the space \mathcal{P}^Δ is more appropriate, a closer study of both extensions can be found in [17, 24].

2.5 Set-valued functions

Definition 2.6 *The graph of a function $g : X \rightarrow \mathcal{P}(Z)$ is defined as*

$$\text{graph } g = \{(x, z) \in X \times Z \mid z \in g(x)\}.$$

If $g : X \rightarrow \mathcal{P}^\Delta$, the domain and epigraph of g are given by

$$\text{dom } g = \{x \in X \mid g(x) \neq \emptyset\}, \quad \text{epi } g = \{(x, z) \in X \times Z \mid z \in g(x) + C\}.$$

If $g : X \rightarrow \mathcal{P}^\Delta$, then $\text{graph } g = \text{epi } g$, motivating the notion *epigraphical type function* for functions $g : X \rightarrow \mathcal{P}^\Delta$.

The set $(\mathcal{P}^\Delta)^X = \{g : X \rightarrow \mathcal{P}^\Delta\}$ equipped with the point-wise addition, multiplication, order relation and inf-residuation is an inf-residuated order complete conlinear space. We denote the inf-convolution of $f, g : X \rightarrow \mathcal{P}^\Delta$ by

$$\forall x \in X : \quad (f \square g)(x) = \inf_{y \in X} (f(x - y) + g(y)) \quad (2.10)$$

and

$$\forall x \in X : \quad (fT)(x) = f(Tx); \quad (Tg)(y) = \inf_{Tx=y} g(x) \quad (2.11)$$

for a linear continuous operator $T : X \rightarrow Y$, $f : Y \rightarrow \mathcal{P}^\Delta$ and $g : X \rightarrow \mathcal{P}^\Delta$. The definitions in (2.10) and (2.11) can be found in [15, 29].

Definition 2.7 A function $g : X \rightarrow \mathcal{P}^\Delta$ is called *positively homogeneous*, iff $\text{epi } g$ is a cone, *convex or closed*, iff $\text{epi } g$ is convex or closed, *subadditive* iff $\text{epi } g$ is closed under addition and *sublinear*, iff $\text{epi } g$ is a convex cone.

Again, a positively homogeneous function is sublinear, if and only if it is subadditive.

If a function $g : X \rightarrow \mathcal{P}^\Delta$ is convex or closed, then especially for each $x \in X$ the set $g(x)$ is convex or closed. Defining $(\text{cl co } g) : X \rightarrow \mathcal{P}^\Delta$ by setting $\text{epi}(\text{cl co } g) = \text{cl co}(\text{epi } g)$, the function $(\text{cl co } g)$ maps into the set \mathcal{G}^Δ of all convex, closed sets $A \in \mathcal{P}(Z)$ with $A = \text{cl co}(A + C)$. Altering the addition of sets to $A \oplus B = \text{cl}(A + B)$ and multiplication with 0 to $0A = \text{cl } C$, then \mathcal{G}^Δ with the altered addition, multiplication and the order relation \supseteq is an inf-residuated order complete conlinear space, the infimum in \mathcal{G}^Δ of a set $\mathcal{A} \subseteq \mathcal{G}^\Delta$ is $\inf_{\mathcal{G}^\Delta} \mathcal{A} = \text{cl co} \bigcup_{A \in \mathcal{A}} A$. This space has been used as image space in [15, 29].

Definition 2.8 A function $g : X \rightarrow \mathcal{P}^\Delta$ is called *proper*, iff $\text{dom } g \neq \emptyset$ and there is no $x \in X$ with $g(x) = Z$. A function g is called z^* -*proper* with $z^* \in Z^*$, iff $x \mapsto -\sigma(z^* | g(x))$ is a proper function.

Each function with nonempty domain is 0-proper and if $g : X \rightarrow \mathcal{P}^\Delta$ is z^* -proper, then $z^* \in C^-$. If a closed convex function $g : X \rightarrow \mathcal{P}^\Delta$ is improper, then $g(x) = Z$ holds for all $x \in \text{dom } g$ and $\text{dom } g$ is a closed convex set in X . Likewise, if a closed convex function is z^* -improper, then $(g(x) - H(z^*)) \setminus (g(x) + H(z^*)) = \emptyset$ is satisfied for all $x \in \text{dom } g$ and $\text{dom } g$ is closed and convex, compare [15, Proposition 5].

3 Scalarization of Set-Valued Functions

Definition 3.1 Let $g : X \rightarrow \mathcal{P}^\Delta$ and $\phi : Z \rightarrow \overline{\mathbb{R}}$ be two functions. The *scalarization* of g with respect to ϕ is defined by

$$\forall x \in X : \quad \varphi_{g,\phi}(x) = \inf \{ -\phi(z) \mid z \in g(x) \}.$$

If $\text{dom } \phi = Z$, then $\text{dom } \varphi_{g,\phi} = \text{dom } g$. Recall that in convex analysis, the indicator function of a set $M \subseteq X$ is the function $I_M : X \rightarrow \overline{\mathbb{R}}$, $I_M(x) = 0$, if $x \in M$ and $I_M(x) = +\infty$, else. If $\phi \in Z^*$, then $\varphi_{g,\phi}(x) = -\sigma(\phi | (g(x)))$, the negative support function of ϕ at $g(x)$ and thus $\varphi_{g,0}(x) = I_{\text{dom } g}(x)$.

If $g(x) = \text{cl co}(g(x)) \in \mathcal{G}^\Delta$, then by Equation (2.6)

$$\forall x \in X : \quad g(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{ z \in Z \mid \varphi_{g,z^*}(x) \leq -z^*(z) \}. \quad (3.1)$$

The scalarization $\varphi_{g,0}$ can be omitted, as $\{ z \in Z \mid \varphi_{g,0}(x) \leq 0 \} = Z$ holds for all $x \in \text{dom } g$ and $\{ z \in Z \mid \varphi_{g,z^*}(x) \leq -z^*(z) \} = \emptyset$ for all $z^* \in C^-$ and $x \notin \text{dom } g$.

A function $g : X \rightarrow \mathcal{G}^\Delta$ is convex, positively homogeneous or subadditive, iff for all $z^* \in C^-$ the scalarization φ_{g,z^*} has the corresponding property. Closedness is not as immediate, as the following example shows. However, if all scalarizations φ_{g,z^*} with $z^* \in C^- \setminus \{0\}$ are closed, then g is closed.

Example 3.2 Let the set $Z = \mathbb{R}^2$ be ordered by the usual ordering cone $C = \mathbb{R}_+^2$, $z^* = (0, -1)$ and $g : \mathbb{R} \rightarrow \mathcal{P}^\Delta$ be defined as $g(x) = \{(\frac{1}{x}, 0)\} + C$, if $x > 0$ and $g(x) = \emptyset$, else. Thus $\text{epi } g$ is a closed set, while $\varphi_{g,z^*}(0) = +\infty$ and $\varphi_{g,z^*}(x) = 0$ holds for all $x > 0$ and therefore $\text{cl } \varphi_{g,z^*}(0) = 0$.

The proof of the following Proposition follows [9, Proposition 3.1].

Proposition 3.3 Let $g : X \rightarrow \mathcal{P}^\Delta$ be a function, then

$$\forall x \in X : \quad (\text{cl co } g)(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \text{cl co } \varphi_{g,z^*}(x) \leq -z^*(z)\}.$$

PROOF. For simplicity suppose that g is a closed convex function. The images $g(x)$ are elements of \mathcal{G}^Δ and because of (3.1) it is left to prove

$$\forall x \in X : \quad g(x) \supseteq \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid \text{cl co } \varphi_{g,z^*}(x) \leq -z^*(z)\}. \quad (3.2)$$

If g is improper, then $\text{dom } g$ is closed and convex, thus $\text{dom } g = \text{dom cl co } \varphi_{g,z^*}$ holds for all $z^* \in C^-$. If $z^* \in C^- \setminus \{0\}$, then $\varphi_{g,z^*}(x) = -\infty$ holds for $x \in \text{dom } g$ and $\varphi_{g,z^*}(x) = +\infty$, else. In this case, (3.2) is immediate.

Suppose g is proper, $(x_0, z_0) \notin \text{epi } g$. By Equation (2.6) it exists $(x^*, z^*) \in X^* \times Z^*$ and $t \in \mathbb{R}$ such that

$$\forall (x, z) \in \text{epi } g : \quad -x^*(x_0) - z^*(z_0) < t < -x^*(x) - z^*(z). \quad (3.3)$$

Thus $z^* \in C^-$ and $x_{-t}^* : X \rightarrow \mathbb{R}$ with $x_{-t}^*(x) = x^*(x) + t$ is an affine minorant of φ_{g,z^*} , separating $(x_0, -z^*(z_0))$ from the epigraph of $\text{cl co } \varphi_{g,z^*}$, proving

$$\forall x \in X : \quad g(x) \supseteq \bigcap_{z^* \in C^-} \{z \in Z \mid \text{cl co } \varphi_{g,z^*}(x) \leq -z^*(z)\}.$$

The strict inequality in formula (3.3) can be satisfied with $z^* = 0$, if and only if $x_0 \notin \text{dom } g$.

Next, chose $(\bar{x}, \bar{z}) \in X \times Z$ such that $\bar{x} \in \text{dom } g$ and $\bar{z} \notin g(\bar{x})$. As $\bar{x} \in \text{dom } g$, it exists $(\bar{x}^*, \bar{z}^*, \bar{t}) \in X^* \times (C^- \setminus \{0\}) \times \mathbb{R}$ with

$$\forall (x, z) \in \text{epi } g : \quad -\bar{x}^*(\bar{x}) - \bar{z}^*(\bar{z}) < \bar{t} < -\bar{x}^*(x) - \bar{z}^*(z).$$

If (\bar{x}^*, \bar{z}^*) separates (x_0, z_0) from $\text{epi } g$, then there is nothing more to prove. Otherwise, we can chose $s > 0$ such that there exists $t_s \in \mathbb{R}$ with

$$-(x^* + s\bar{x}^*)(x_0) - (z^* + s\bar{z}^*)(z_0) < t_s < -(x^* + s\bar{x}^*)(x) - (z^* + s\bar{z}^*)(z)$$

for all $(x, z) \in \text{epi } g$. By assumption $(z^* + s\bar{z}^*) \in C^- \setminus \{0\}$ is fulfilled and $(x^* + s\bar{x}^*)_{-t_s}$ separates $(x_0, -(z^* + s\bar{z}^*)(z_0))$ from $\text{cl co } (\text{epi } \varphi_{g,(z^* + s\bar{z}^*)})$, thus $z_0 \notin \{z \in Z \mid \text{cl co } \varphi_{g,(z^* + s\bar{z}^*)}(x_0) \leq -(z^* + s\bar{z}^*)(z)\}$ and (3.2) is proven. \square

As an immediate corollary we get

Corollary 3.4 *Let $f, g : X \rightarrow \mathcal{P}^\Delta$ be two functions. Then $(\text{clco } f) \leq (\text{clco } g)$ is satisfied, iff for all $z^* \in C^- \setminus \{0\}$ the function $(\text{clco } \varphi_{f,z^*})$ is a minorant of $(\text{clco } \varphi_{g,z^*})$.*

Moreover, $\text{clco } g : X \rightarrow \mathcal{P}^\Delta$ is either proper or constant Z if and only if the following equality is satisfied for all $x \in X$.

$$(\text{clco } g)(x) = \bigcap_{\substack{\text{clco } \varphi_{g,z^*} \text{ is proper,} \\ z^* \in C^- \setminus \{0\}}} \{z \in Z \mid \text{clco } \varphi_{g,z^*}(x) \leq -z^*(z)\}. \quad (3.4)$$

Especially, $g : X \rightarrow \mathcal{P}^\Delta$ is proper if and only if it is z^* -proper for some $z^* \neq 0$.

Proposition 3.5 *Let I be an index set, $f, g, g_i : X \rightarrow \mathcal{P}^\Delta$ functions for all $i \in I$, $T : X \rightarrow Y$ a linear continuous operator and $h : Y \rightarrow \mathcal{P}^\Delta$ a function. Let $z^* \in C^-$, then the following formulas are true.*

- (a) $\forall x \in X : \quad \varphi_{f+g,z^*}(x) = \varphi_{f,z^*}(x) \dot{+} \varphi_{g,z^*}(x).$
- (b) $\forall x \in X : \quad \varphi_{hT,z^*}(x) = \varphi_{h,z^*}(T(x)).$
- (c) $\forall x \in X : \quad \varphi_{\inf_{i \in I} g_i, z^*}(x) = \inf_{i \in I} \varphi_{g_i, z^*}(x).$

PROOF. (a) and (b) are immediate from the Definition 3.1. By definition $(\inf_{i \in I} g_i)(x) = \bigcup_{i \in I} g_i(x)$ for all $x \in X$. Thus, (c) is immediate. \square

Combining (a) and (c) from Proposition 3.5 one can derive

$$\forall x \in X : \quad \varphi_{f \square g, z^*}(x) = (\varphi_{f, z^*} \square \varphi_{g, z^*})(x) \quad (3.5)$$

and by (b) and (c)

$$\forall x \in X : \quad \varphi_{Th, z^*}(x) = T \varphi_{h, z^*}(x). \quad (3.6)$$

Proposition 3.6 [19] *Let $f, g : X \rightarrow \mathcal{P}^\Delta$ and $g_i : X \rightarrow \mathcal{P}^\Delta$ be functions for all $i \in I$ and $x \in X$ and $z^* \in C^-$, then*

$$\begin{aligned} \varphi_{f, z^*}(x) \dot{-} \varphi_{g, z^*}(x) &\leq \varphi_{(f \dot{-} g), z^*}(x); \\ \sup_{i \in I} \varphi_{g_i, z^*}(x) &\leq \varphi_{\sup_{i \in I} g_i, z^*}(x). \end{aligned}$$

If additionally $f(x) = H_{\varphi_{f, z^}(x)}(z^*)$ holds true or $\sup_{i \in I} g_i(x) = \bigcap_{i \in I} H_{\varphi_{g_i, z^*}(x)}(z^*)$ respectively, then we get equality.*

If $c : (\mathcal{P}^\Delta)^X \rightarrow (\mathcal{P}^\Delta)^Y$ is a duality in the sense of Singer, [30] and $c(g)(x) = H_{\varphi_{c(g), z^*}(x)}(z^*)$ holds true for all $g : X \rightarrow \mathcal{P}^\Delta$ then by applying Proposition 3.5 (c) and Proposition 3.6 the following mapping is a duality:

$$\forall g : X \rightarrow \mathcal{P}^\Delta, \forall z^* \in C^- : \quad \varphi_{g, z^*} \mapsto \varphi_{c(g), z^*}.$$

Definition 3.7 Let $f : X \rightarrow \overline{\mathbb{R}}$ and $\phi : Z \rightarrow \overline{\mathbb{R}}$ be two functions. The set-valued function $S_{(f,\phi)} : X \rightarrow \mathcal{P}(Z)$ is defined by

$$\forall x \in X : S_{(f,\phi)}(x) = \{z \in Z \mid f(x) \leq -\phi(z)\}.$$

A function f_1 is a minorant of $f_2 : X \rightarrow \overline{\mathbb{R}}$, iff $S_{(f_1,\phi)}(x) \supseteq S_{(f_2,\phi)}(x)$ is met for all $x \in X$ and all $\phi : Z \rightarrow \overline{\mathbb{R}}$.

Each function $f : X \rightarrow \overline{\mathbb{R}}$ is dominated by $\varphi_{S_{(f,\phi)},\phi}$. If $\phi(X) \supseteq \mathbb{R}$, then equality holds true.

If $\phi : Z \rightarrow \overline{\mathbb{R}}$ is nonincreasing, i.e. if $z_1 \leq_C z_2$, then $\phi(z_2) \leq \phi(z_1)$, then $S_{(f,\phi)}$ takes its values in \mathcal{P}^Δ for all $f : X \rightarrow \overline{\mathbb{R}}$. If additionally $\phi = z^* \in C^-$, then the values of $S_{(f,z^*)}$ are of the form $S_{(f,z^*)}(x) = H_{f(x)}(z^*)$, i.e. affine half-spaces or \emptyset or Z , the image-space is a subset of $\mathcal{P}^\Delta(Z, H(z^*))$.

Proposition 3.8 Let $f : X \rightarrow \overline{\mathbb{R}}$, $z^* \in C^-$ and $g : X \rightarrow \mathcal{P}^\Delta$. Then $S_{(f,z^*)}$ is a minorant of g if and only if f is a minorant of φ_{g,z^*} .

PROOF. First, let $S_{(f,z^*)}(x) \supseteq g(x)$ be assumed for all $x \in X$. Then

$$\forall x \in X : f(x) \leq \varphi_{S_{(f,z^*)},z^*}(x) \leq \varphi_{g,z^*}(x)$$

is satisfied, f is a minorant of φ_{g,z^*} . On the other hand, if f is dominated by φ_{g,z^*} , then

$$\forall x \in X : S_{(f,z^*)}(x) \supseteq S_{(\varphi_{g,z^*},z^*)}(x) \supseteq g(x).$$

□

Proposition 3.9 Let $g : X \rightarrow \mathcal{P}^\Delta$ be a function, $z^* \in C^-$. It holds

$$\begin{aligned} \forall x \in X : S_{(\varphi_{g,z^*},z^*)}(x) &= \text{cl } (g(x) + H_0(z^*)); \\ \text{cl co } (g(x)) &= \bigcap_{z^* \in C^- \setminus \{0\}} S_{(\varphi_{g,z^*},z^*)}(x). \end{aligned}$$

PROOF. By definition,

$$\varphi_{g,z^*}(x) = -\sigma(z^* | g(x)), \quad S_{(\varphi_{g,z^*},z^*)}(x) = H_{0+}(-\sigma(z^* | g(x)))(z^*),$$

thus by Lemma 2.5 the first equation is proven while the second is proven by a separation argument in Z , compare Equation (2.6). □

Corollary 3.10 Let $f : X \rightarrow \overline{\mathbb{R}}$ be a function $x \in X$. Then

$$\begin{aligned} f(x) \leq 0 &\Leftrightarrow S_{(f,0)}(x) = Z &\Leftrightarrow \varphi_{S_{(f,0)},0}(x) = 0; \\ 0 < f(x) &\Leftrightarrow S_{(f,0)}(x) = \emptyset &\Leftrightarrow \varphi_{S_{(f,0)},0}(x) = +\infty. \end{aligned}$$

The scalarization $\varphi_{S_{(f,0)},0}$ of $f : X \rightarrow \overline{\mathbb{R}}$ is the indicator function of the sublevel set of f at 0.

Example 3.11 Let $x^* \in X^*$ and $r \in \mathbb{R}$. The function $x_r^* : X \rightarrow \mathbb{R}$ is defined by $x_r^*(x) = x^*(x) - r$ for all $x \in X$, the closed improper inf-extension $\hat{x}_r^* : X \rightarrow \mathbb{R}^\Delta$ by

$$\hat{x}_r^*(x) = \begin{cases} -\infty & : x_r^*(x) \leq 0 \\ +\infty & : x_r^*(x) > 0 \end{cases}$$

If $x^* = 0$, then we obtain $\hat{x}_r^* = -\infty$ if $r \geq 0$, and $+\infty$ else. If $f : X \rightarrow \mathbb{R}^\Delta$ is a function, then

$$\forall x \in X : \quad \hat{x}_r^*(x) \leq f(x) \quad \Leftrightarrow \quad x^*(x) - r \leq I_{\text{dom } f}(x). \quad (3.7)$$

We denote

$$\forall z^* \in C^-, \forall r \in \mathbb{R}, \forall x \in X : \quad S_{(x^*, z^*, r)}(x) = S_{(x_r^*, z^*)}(x)$$

and for completeness

$$\forall z^* \in C^-, \forall x \in X : \quad S_{(x^*, z^*, +\infty)}(x) = Z, \quad S_{(x^*, z^*, -\infty)}(x) = \emptyset.$$

Thus, $S_{(x^*, z^*)}(x) = S_{(x^*, z^*, 0)}(x)$ and for $(x^*, r) \in X^* \times \mathbb{R}$ and $z^* \in C^-$ it holds

$$\forall x \in X : \quad S_{(\hat{x}_r^*, z^*)}(x) = S_{(x_r^*, 0)}(x) = \begin{cases} Z, & \text{if } x_r^*(x) \leq 0; \\ \emptyset, & \text{else.} \end{cases}$$

The scalarization $\varphi_{S_{(x_r^*, 0), 0}}$ is the indicator function of $\text{dom } \hat{x}_r^*$, while $\hat{x}_r^* = \varphi_{S_{(\hat{x}_r^*, z^*), z^*}}$ is satisfied if $z^* \neq 0$.

A function of the type $S_{(x^*, z^*, r)} : X \rightarrow \mathcal{P}^\Delta$ with $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$ is called conaffine. If additionally $r = 0$, then $S_{(x^*, z^*)} : X \rightarrow \mathcal{P}^\Delta$ is called conlinear. If $z^* \neq 0$, then $S_{(x^*, z^*, r)}$ is proper and its values are affine half-spaces, if $z^* = 0$, then $S_{(x^*, 0, r)}(x) = Z$ for all $x \in \text{dom } \hat{x}_r^*$ and $S_{(x^*, 0, r)}(x) = \emptyset$ for $x \notin \text{dom } \hat{x}_r^*$.

Proposition 3.12 Let $g : X \rightarrow \mathcal{P}^\Delta$, $z^* \in C^- \setminus \{0\}$, $x^* \in X^*$ and $r \in \mathbb{R}$. Then x_r^* is a minorant of φ_{g, z^*} if and only if $S_{(x^*, z^*, r)}$ is a minorant of g . The closed convex hull of g is proper or constant \emptyset or Z , if and only if

$$\forall x \in X : \quad (\text{cl co } g)(x) = \bigcap_{\substack{x_r^* \leq \varphi_{g, z^*}, \\ z^* \in C^- \setminus \{0\}}} S_{(x^*, z^*, r)}(x),$$

it is improper, iff

$$\forall x \in X : \quad (\text{cl co } g)(x) = \bigcap_{x_r^* \leq \varphi_{g, 0}} S_{(x^*, 0, r)}(x).$$

PROOF. By Proposition 3.8 x_r^* is a minorant of φ_{g, z^*} if and only if $S_{(x^*, z^*, r)}$ is a minorant of g and by Corollary 3.4

$$\forall x \in X : \quad (\text{cl co } g)(x) = \bigcap_{z^* \in C^- \setminus \{0\}} S_{(\text{cl co } \varphi_{g, z^*}, z^*)}(x)$$

is satisfied. The closed convex hull of a scalar function $\varphi_{g,z^*} : X \rightarrow \mathbb{R}^\Delta$ is the supremum of its affine minorants if and only if $\text{cl co } \varphi_{g,z^*}$ is proper or constant $+\infty$ or $-\infty$. In this case,

$$\forall x \in X : \quad S_{(\text{cl co } \varphi_{g,z^*}, z^*)}(x) = \bigcap_{x_r^* \leq \varphi_{g,z^*}} S_{(x_r^*, z^*)}(x).$$

On the other hand, $\text{cl co } \varphi_{g,z^*}$ is not proper for some $z^* \in C^- \setminus \{0\}$, iff either $\varphi_{g,z^*} \equiv +\infty$, thus $g \equiv \emptyset$, or φ_{g,z^*} does not have any affine minorants, proving the first statement.

The function $\text{cl co } g : X \rightarrow \mathcal{P}^\Delta$ is improper, iff $(\text{cl co } g)(x) = Z$ for all $x \in \text{cl co dom } g = \text{dom}(\text{cl co } g)$, iff for each $z^* \in C^- \setminus \{0\}$ the scalarization φ_{g,z^*} is improper. It holds

$$\forall x \in X : \quad \hat{x}_r^*(x) \leq \varphi_{g,z^*}(x) \quad \Leftrightarrow \quad x_r^*(x) \leq I_{\text{dom } \varphi_{g,z^*}}(x) = \varphi_{g,0}(x),$$

compare (3.7). Thus

$$\forall x \in X : \quad x_r^*(x) \leq \varphi_{g,0}(x) \quad \Leftrightarrow \quad S_{(x^*, 0, r)}(x) \supseteq g(x)$$

and

$$\bigcap_{x_r^* \leq \varphi_{g,0}} S_{(x^*, 0, r)}(x) = \begin{cases} Z, & \text{if } x \in \text{cl co dom } g \\ \emptyset, & \text{else} \end{cases}$$

proving the second statement. \square

Especially,

$$\forall x \in X : \quad S_{(x^*, 0, r)}(x) \supseteq g(x) \quad \Leftrightarrow \quad x^*(x) \dot{-} r \leq I_{\text{dom } g}(x).$$

Notice that the scalarizations of $g : X \rightarrow \mathcal{P}^\Delta$ used in Proposition 3.12 are either proper or constant $+\infty$ and thus also the affine minorants needed in the representation are proper. Alternatively, the representation can be done excluding the 0-scalarization $\varphi_{g,0}$, compare [19] but at the cost of properness of the scalarizations and thus also of the affine minorants. The same is true for the following theorem, which is a corollary of Proposition 3.12 and Lemma 2.1.

Theorem 3.13 *Let $g : X \rightarrow \mathcal{P}^\Delta$ be a function, then g is convex and closed, if and only if g is the point-wise supremum of its conaffine minorants. The function g is proper or constant \emptyset or Z , if and only if it is the point-wise supremum of its proper conaffine minorants,*

$$\forall x \in X : \quad g(x) = \bigcap_{\substack{S_{(x^*, z^*, r)} \preccurlyeq_C g, \\ z^* \in C^- \setminus \{0\}}} S_{(x^*, z^*, r)}(x). \quad (3.8)$$

Otherwise, g is the point-wise supremum of its improper conaffine minorants,

$$\forall x \in X : \quad g(x) = \bigcap_{S_{(x^*, 0, r)} \preccurlyeq_C g} S_{(x^*, 0, r)}(x). \quad (3.9)$$

Part of the statement of Theorem 3.13 has been proven for the proper case in [15, Theorem 1], while in [19, Theorem 5.30] a representation formula with improper conaffine minorants is proven.

Example 3.14 [15, Proposition 8] Let $\bar{g} : X \rightarrow Z$ be a single-valued function with the epigraphical extension $g(x) = \{\bar{g}(x)\} + C$ for all $x \in X$, C a closed convex cone and $T : X \rightarrow Z$ is a linear continuous operator. Then

$$\begin{aligned} \forall x \in X : \quad \varphi_{g,z^*}(x) &= -z^*(\bar{g}(x)); \\ S_{(-T^*z^*, z^*)}(x) &= \{Tx\} + H(z^*) \end{aligned}$$

is satisfied for all $z^* \in C^- \setminus \{0\}$. Moreover, $T(x) + z_0 \leq \bar{g}(x)$ is met for all $x \in X$ if and only if

$$\forall z^* \in C^- \setminus \{0\}, \forall x \in X : \quad S_{(-T^*z^*, z^*)}(x) + \{z_0\} \supseteq g(x).$$

Remark 3.15 In [1], the case of an ordering cone with compact base and nonempty interior is considered. The authors identify a vector-valued function $V_{f,K} : X \rightarrow Z$ with respect to a total ordering cone K . Under some additional assumptions,

$$-z^*(V_{f,K}(x)) = \varphi_{f,z^*}(x)$$

for all $z^* \in K^-$ and $V_{f,K} \in f(x)$. However, the existence of such a minimal element is only given under strong assumptions on both the ordering cone and the function itself.

4 Conjugation of Set-Valued Functions

Definition 4.1 The conjugate of a function $g : X \rightarrow \mathcal{P}^\Delta$ is $g^* : X^* \times C^- \times \mathbb{R} \rightarrow \mathcal{P}^\Delta$, defined by

$$\forall (x^*, z^*, r) \in X^* \times C^- \times \mathbb{R} : \quad g^*(x^*, z^*, r) = \sup_{x \in X} (S_{(x^*, z^*, r)}(x) \dot{-} g(x)).$$

For completeness we define

$$\begin{aligned} g^*(x^*, z^*, -\infty) &= \sup_{x \in X} (S_{(x^*, z^*, -\infty)}(x) \dot{-} g(x)) = \begin{cases} Z, & \text{if } \text{dom } g = \emptyset, \\ \emptyset, & \text{else,} \end{cases} \\ g^*(x^*, z^*, +\infty) &= \sup_{x \in X} (S_{(x^*, z^*, +\infty)}(x) \dot{-} g(x)) = Z. \end{aligned}$$

Especially, the values of g^* are affine half-spaces or \emptyset or Z . With $(\varphi_{g,z^*})^* : X^* \times \mathbb{R} \rightarrow \overline{\mathbb{R}}$ we denote

$$(\varphi_{g,z^*})^*(x^*, r) = \sup_{x \in X} (x_r^*(x) \dot{-} \varphi_{g,z^*}(x)),$$

compare [19] and abbreviate $g^*(x^*, z^*) = g^*(x^*, z^*, 0)$ and $(\varphi_{g,z^*})^*(x^*) = (\varphi_{g,z^*})^*(x^*, 0)$ for all $(x^*, z^*) \in X^* \times C^-$.

Proposition 4.2 Let $g : X \rightarrow \mathcal{P}^\Delta$ be a function and $x^* \in X^*$, $z^* \in C^-$ and $r \in \mathbb{R}$, then

$$\begin{aligned} (\varphi_{g,z^*})^*(x^*, r) &= (\varphi_{g,z^*})^*(x^*) \dot{-} r, \\ g^*(x^*, z^*, r) &= H_{(\varphi_{g,z^*})^*(x^*, r)}(z^*) \end{aligned}$$

is satisfied. If $z^* \in C^- \setminus \{0\}$, then additionally

$$\begin{aligned}(\varphi_{g,z^*})^*(x^*, r) &= \varphi_{g^*(\cdot, z^*, \cdot), z^*}(x^*, r), \\ g^*(x^*, z^*, r) &= g^*(x^*, z^*) \dot{-} H_r(z^*)\end{aligned}$$

holds true while

$$\begin{aligned}(\varphi_{g,0})^*(x^*, r) &= \sigma(x^* | \text{dom } g) \dot{-} r, \\ \varphi_{g^*(\cdot, 0, \cdot), 0}(x^*, r) &= I_{\text{dom } g^*(\cdot, 0, \cdot)}(x^*, r)\end{aligned}$$

and

$$g^*(x^*, 0, r) = \begin{cases} Z, & \text{if } \sigma(x^*, \text{dom } g) \leq r \\ \emptyset, & \text{else} \end{cases}$$

is satisfied.

PROOF. The equation $(\varphi_{g,z^*})^*(x^*, r) = (\varphi_{g,z^*})^*(x^*) \dot{-} r$ is proven by direct calculation, as by assumption $r \in \mathbb{R}$. It is well known and easy to prove that $(I_{\text{dom } g})^*(x^*) = \sigma(x^* | \text{dom } g)$, hence

$$(\varphi_{g,0})^*(x^*, r) = (I_{\text{dom } g})^*(x^*, r) = \sigma(x^* | \text{dom } g) \dot{-} r.$$

Applying Lemma 2.5 and Proposition 3.6 gives

$$g^*(x^*, z^*, r) = \bigcap_{x \in X} H_{x_r^*(x) \dot{-} \varphi_{g,z^*}(x)}(z^*) = H_{(\varphi_{g,z^*})^*(x^*, r)}(z^*)$$

and thus

$$\varphi_{g^*(\cdot, z^*, \cdot), z^*}(x^*, r) = \inf \{ -z^*(z) \mid (\varphi_{g,z^*})^*(x^*, r) \leq -z^*(z) \}.$$

If additionally $z^* \neq 0$, then

$$\varphi_{g^*(\cdot, z^*, \cdot), z^*}(x^*, r) = \inf \{ -z^*(z) \mid (\varphi_{g,z^*})^*(x^*, r) \leq -z^*(z) \} = (\varphi_{g,z^*})^*(x^*, r)$$

is satisfied while

$$g^*(x^*, 0, r) = H_{\sigma(x^*, \text{dom } g) \dot{-} r}(0) = \begin{cases} Z, & \text{if } \sigma(x^*, \text{dom } g) \leq r \\ \emptyset, & \text{else.} \end{cases}$$

and

$$\begin{aligned} \varphi_{g^*(\cdot, 0, \cdot), 0}(x^*, r) &= \inf \{ 0 \mid (\varphi_{g,z^*})^*(x^*, r) \leq 0 \} \\ &= \begin{cases} 0, & \text{if } \sigma(x^* | \text{dom } g) \dot{-} r \leq 0 \\ +\infty, & \text{else} \end{cases} \\ &= I_{\text{dom } g^*(\cdot, 0, \cdot)}(x^*, r). \end{aligned}$$

For all $z^* \in C^- \setminus \{0\}$ and all $a, b \in \overline{\mathbb{R}}$, by Lemma 2.5 it holds

$$H_{a \dot{-} b}(z^*) = \{ z \in Z \mid a \leq -z^*(z) + b \} = H_a(z^*) \dot{-} H_b(z^*).$$

Thus especially $g^*(x^*, z^*, r) = g^*(x^*, z^*) - H_r(z^*)$ is true, proving the statement. \square

Compare Corollary 3.10 for the scalarization with $z^* = 0$. In Section 5, set-valued duality results will be proven which only hold true on $X^* \times \{0\} \subseteq X^* \times C^-$, if the pre-image space of the conjugate function is the set of conaffine, rather than conlinear functions. For this reason, we define the conjugate of a set-valued function to map from $X^* \times C^- \times \mathbb{R}$, rather than just $X^* \times C^-$ to \mathcal{P}^Δ .

Proposition 4.3 *Let $g : X \rightarrow \mathcal{P}^\Delta$ be a function, then the conjugate of g and the conjugate of the closed convex hull of g coincide,*

$$\forall x^* \in X^*, \forall z^* \in C^-, \forall r \in \mathbb{R} : \quad (\text{cl co } g)^*(x^*, z^*, r) = g^*(x^*, z^*, r).$$

PROOF. By Proposition 3.3, $(\text{cl co } \varphi_{g, z^*})(x) \leq (\text{cl co } \varphi_{\text{cl co } g, z^*})(x)$, thus

$$\forall x \in X, \forall z^* \in C^- : \quad \text{cl co } \varphi_{g, z^*}(x) = \text{cl co } \varphi_{\text{cl co } g, z^*}(x) \quad (4.1)$$

and as

$$(\text{cl co } g)^*(x^*, z^*, r) = \{z \in Z \mid (\varphi_{\text{cl co } g, z^*})^*(x^*) \leq r - z^*(z)\}$$

is satisfied for all $x^* \in X^*$, $r \in \mathbb{R}$ and all $z^* \in C^-$ and we can conclude $(\text{cl co } g)^*(x^*, z^*, r) = g^*(x^*, z^*, r)$. \square

In contrast to the present approach, the (negative) conjugate in [15] is defined as a G^Δ -valued function via an infimum rather than a supremum and thus avoiding a difference operation on the power set $\mathcal{P}(Z)$. In [29] and in [19], the same idea as in Definition 4.1 has been used. In [29], the dual variables are reduced to the set $X^* \times C^- \setminus \{0\}$, while in [19] the dual space is the set of all conaffine functions and again, $z^* = 0$ is prohibited. There, improper scalarizations play an important role while in the present approach we avoid those at the expense of including $z^* = 0$ in the dual space.

Definition 4.4 *To a function $g : X \rightarrow \mathcal{P}^\Delta$, the biconjugate $g^{**} : X \rightarrow \mathcal{P}^\Delta$ is defined by*

$$\forall x \in X : \quad g^{**}(x) = \bigcap_{(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}} (S_{(x^*, z^*, r)}(x) - g^*(x^*, z^*, r)). \quad (4.2)$$

It can be proven easily that

$$\sup_{(x^*, r) \in X^* \times \mathbb{R}} (x_r^*(x) - (\varphi_{g, z^*})^*(x^*, r)) = \sup_{x^* \in X^*} (x^*(x) - (\varphi_{g, z^*})^*(x^*))$$

thus the biconjugate of a scalar function can be defined as usual, setting

$$(\varphi_{g, z^*})^{**}(x) = \sup_{x^* \in X^*} (x^*(x) - (\varphi_{g, z^*})^*(x^*)).$$

Theorem 4.5 (Biconjugation Theorem) *Let $g : X \rightarrow \mathcal{P}^\Delta$, then*

$$\forall x \in X : \quad (\text{cl co } g)(x) = g^{**}(x) = \bigcap_{z^* \in C^-} \{z \in Z \mid (\varphi_{g, z^*})^{**}(x) \leq -z^*(z)\} \quad (4.3)$$

and $\text{cl co } g$ is proper or constant \emptyset or Z , if and only if equality is satisfied when omitting $z^* = 0$:

$$\forall x \in X : \quad g^{**}(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid (\varphi_{g,z^*})^{**}(x) \leq -z^*(z)\}. \quad (4.4)$$

PROOF. By Proposition 4.2 the conjugate of a function is represented by $g^*(x^*, z^*, r) = H_{(\varphi_{g,z^*})^*(x^*) \rightarrow r}(z^*)$ for all $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$, thus by Definition 4.4 we conclude

$$g^{**}(x) = \bigcap_{z^* \in C^-} \left(\bigcap_{(x^*, r) \in X^* \times \mathbb{R}} \left(H_{x^*(x) \rightarrow r}(z^*) \rightarrow H_{(\varphi_{g,z^*})^*(x^*) \rightarrow r}(z^*) \right) \right)$$

for all $x \in X$. Thus

$$\begin{aligned} g^{**}(x) &= \bigcap_{z^* \in C^-} H_{\sup_{x^* \in X^*} (x^*(x) \rightarrow (\varphi_{g,z^*})^*(x^*))}(z^*) \\ &= \bigcap_{z^* \in C^-} \{z \in Z \mid \varphi_{g,z^*}^{**}(x) \leq -z^*(z)\} \end{aligned}$$

is fulfilled for all $x \in X$, $\varphi_{g,0}^{**}(x) = I_{\text{cl co dom } g}(x)$ and

$$(\text{cl co } g)(x) = \bigcap_{z^* \in C^-} \{z \in Z \mid (\text{cl co } \varphi_{g,z^*})(x) \leq -z^*(z)\}.$$

By the scalar biconjugation theorem, $(\text{cl co } \varphi_{g,z^*})(x) = \varphi_{g,z^*}^{**}(x)$ is met for all $x \in \text{cl co dom } g$ and $\text{cl co } \varphi_{g,z^*} = \varphi_{g,z^*}^{**}$ holds true iff $\text{cl co } \varphi_{g,z^*}$ is either proper or constant $+\infty$ or $-\infty$, thus $(\text{cl co } g)(x) = g^{**}(x)$ for all $x \in X$. If $\text{cl co } g$ is proper or constant Z or \emptyset , then either $\text{cl co } \varphi_{g,z^*}(x) = \varphi_{g,z^*}^{**}(x) = +\infty$ for all $x \in X$ or by Corollary 3.4

$$(\text{cl co } g)(x) = \bigcap_{\substack{\text{cl co } \varphi_{g,z^*} \text{ is proper,} \\ z^* \in C^- \setminus \{0\}}} \{z \in Z \mid (\varphi_{g,z^*})^{**}(x) \leq -z^*(z)\}.$$

Moreover, $(\varphi_{g,z^*})^{**}$ is constant $-\infty$, whenever $\text{cl co } (\varphi_{g,z^*})$ is an improper function with $\text{dom } \varphi_{g,z^*} \neq \emptyset$, hence in this case

$$\forall x \in X : \quad g^{**}(x) = \bigcap_{z^* \in C^- \setminus \{0\}} \{z \in Z \mid (\varphi_{g,z^*})^{**}(x) \leq -z^*(z)\}$$

is satisfied. Finally, if (4.4) is met and $g^{**}(x) = Z$ for some $x \in X$, then

$$\forall z^* \in C^- \setminus \{0\} : (\varphi_{g,z^*})^{**}(x) = -\infty.$$

In this case, g^{**} is constant Z , proving the statement. \square

Assuming the order cone C to be closed and pointed, a conjugate of a vector-valued function $f : X \rightarrow Z$ is defined in [4, 32] and the references therein. The pre-image space of

the conjugate is the set of continuous linear operators $T : X \rightarrow Z$, the conjugate is defined by

$$f^+(T) = \sup_{x \in X} (T(x) - f(x)).$$

To guarantee the existence of $f^+(T)$, the order induced by C is assumed to fulfill a least upper bound property [32] or even order completeness, [4]. Identifying $f_C(x) = f(x) + C$, the following representation is fulfilled.

$$\begin{aligned} f^+(T) + H(z^*) &= (f_C)^*(-T^*z^*, z^*), \\ f^+(T) + \text{cl } C &= \bigcap_{z^* \in C^-} (f_C)^*(-T^*z^*, z^*). \end{aligned}$$

Thus, results on the conjugate f^+ are included in the more general results on our set-valued conjugate.

The reader is referred to [15, Proposition 12, 13; Theorem 2, 3], [29, Section 4], [12, Corollary 4.2] for a more thorough investigation of the conjugates.

Theorem 4.5 is a set-valued Fenchel–Moreau theorem, including the improper case alongside to the proper case. The proper case can be found in [15, Theorem 2] or in [29, Theorem 4.1.15].

5 Duality Results

In analogy to the scalar case, a chain-rule as well as a Sandwich Theorem and the Fenchel–Rockafellar Duality Theorem can be proven for set-valued functions. We abbreviate the proofs by citing scalar results and applying Proposition 4.2 and Theorem 4.5. Direct proofs for a special case can be found in [15, 16]. There, strong duality results are formulated under the additional assumption of an inner point $(x_0, z_0) \in \text{int epi } g$ and $z^* \neq 0$. We will show that continuity of g in x_0 in the sense of [2, 13], too, is a sufficient assumption for strong duality results.

Proposition 5.1 *Let $z^* \in C^-$ and $g : X \rightarrow \mathcal{P}^\Delta$ a convex function, $x_0 \in \text{dom } g$. If either of the following conditions is met, then φ_{g,z^*} is convex and either continuous in x_0 or $\varphi_{g,z^*}(x_0 + x) = -\infty$ is satisfied for all elements x of an open subset $V \subseteq X$ with $0 \in V$.*

- (a) *g is lower continuous in $x_0 \in \text{dom } g$ in the sense of [13, Definition 2.5.1.], i.e. for all open sets $D \subseteq Z$ with $g(x_0) \cap D \neq \emptyset$ there exists a 0-neighborhood $V \subseteq X$ such that*

$$\forall x \in V : \quad g(x_0 + x) \cap D \neq \emptyset; \tag{5.1}$$

- (b) *there is $z_0 \in g(x_0)$ such that $\{x \in X \mid z_0 \in g(x)\}$ is a neighborhood of x_0 .*

PROOF. As $g : X \rightarrow \mathcal{P}^\Delta$ is by assumption convex, so is each scalarization φ_{g,z^*} .

- (a) To $z^* \in C^-$ and $t \in \overline{\mathbb{R}}$, define the open set $S_t(z^*) = \{z \in Z \mid t < -z^*(z)\} \subseteq Z$ and assume g to be lower continuous in $x_0 \in \text{dom } g$ and $\varepsilon > 0$.

If $z^* = 0$, then $g(x_0) \cap S_{-\varepsilon}(z^*) \neq \emptyset$ and thus it exists a 0-neighborhood $V \subseteq X$ such that $g(x_0 + x) \cap S_{-\varepsilon}(z^*) \neq \emptyset$ is satisfied for all $x \in V$. Therefore, $\varphi_{g,0} = I_{\text{dom } g}$ is continuous at x_0 .

If $z^* \in C^- \setminus \{0\}$, then $g(x_0) \cap S_{\varphi_{g,z^*}(x_0)+\varepsilon}(z^*) \neq \emptyset$ and it exists a 0-neighborhood $V \subseteq X$ such that φ_{g,z^*} is bounded from above on the set $\{x_0\} + V$ by $(\varphi_{g,z^*}(x_0) + \varepsilon)$ and thus φ_{g,z^*} is either continuous in x_0 or $\varphi_{g,z^*}(x_0 + x) = -\infty$ for all $x \in V$.

- (b) Let g be convex and $z_0 \in g(x_0)$ such that $N = \{x \in X \mid z_0 \in g(x)\}$ is a neighborhood of x_0 , then φ_{g,z^*} is bounded from above on the set N by $\varphi_{g,z^*}(x) \leq -z^*(z_0)$ for all $x \in N$. Thus φ_{g,z^*} is either continuous in x_0 or $\varphi_{g,z^*}(x_0 + x) = -\infty$ for all elements x of an open set $V \subseteq X$ with $0 \in V$. \square

It is easy to check that under the assumptions of Proposition 5.1 each scalarization satisfies $\varphi_{g,z^*}(x_0) = (\text{cl co } \varphi_{g,z^*})(x_0)$. Hence the assumptions of Proposition 5.1 are sufficient for the following two equalities

$$g(x_0) = (\text{cl co } g)(x_0) = \text{cl co } (g(x_0)) \quad (5.2)$$

A more thorough investigation on continuity notions and sufficient constrained qualifications for strong duality in set-valued optimization will be done in the forthcoming work [20].

In the proof of Proposition 5.1, we use Property (5.1) for the open half-spaces $S_t(z^*)$ with $t \in \overline{\mathbb{R}}$ and $z^* \in C^-$. In case Z is equipped with a norm, this leads to a uniform structure studied in [31], compare also [27, 26]. Under the associated topology τ_S on Z , a sequence of closed convex sets $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{G}^\Delta$ converges towards $A \in \mathcal{G}^\Delta$ if and only if for all $z^* \in C^-$, $\liminf_{n \rightarrow \infty} \{-z^*(a_n) \mid a_n \in A_n\} = \inf \{-z^*(a) \mid a \in A\}$ is satisfied. Thus, if a convex function $g : X \rightarrow (\mathcal{G}^\Delta, \tau_S)$ is continuous in $x_0 \in X$ and z^* -proper, then especially $\varphi_{g,z^*} : X \rightarrow \mathbb{R}^\Delta$ is continuous. Also, if $g : X \rightarrow \mathcal{G}^\Delta$ is continuous in the sense of [13] and Z is equipped with a norm, then g is continuous under the topology τ_S .

Definition 5.2 Let $g_1, g_2 : X \rightarrow \mathcal{P}^\Delta$ be two functions, $f : X \rightarrow \mathcal{P}^\Delta$ a function, $z^* \in C^-$ and $T : X \rightarrow Y$, $S : Y \rightarrow X$ linear continuous operators.

- (a) Define the infimal convolution of g_1^* and g_2^* in $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$ with respect to $+$ by

$$(g_1^* \square g_2^*)(x^*, z^*) = \text{cl} \bigcup_{\substack{x_1^*, x_2^* \in X^*, x_1^* + x_2^* = x^* \\ r_1, r_2 \in \mathbb{R}, r_1 + r_2 = r}} (g_1^*(x_1^*, z^*, r_1) + g_2^*(x_2^*, z^*, r_2)).$$

- (b) Define

$$\begin{aligned} \forall (x^*, r) \in X^* \times \mathbb{R} : \quad (T^* f^*)(x^*, z^*, r) &= \text{cl} \bigcup_{T^* y^* = x^*} f^*(y^*, z^*, r) \\ (f^* S^*)(x^*, z^*, r) &= f^*(S^* x^*, z^*, r). \end{aligned}$$

As each image $(g_1^* \square g_2^*)(x^*, z^*)$ and $(T^* f^*)(x^*, z^*, r)$ is by definition a closed half-space, we obtain

$$\begin{aligned} (g_1^* \square g_2^*)(x^*, z^*) &= H_{-\sigma(z^* | (g_1^* \square g_2^*)(x^*, z^*))}(z^*); \\ (T^* f^*)(x^*, z^*, r) &= H_{-\sigma(z^* | (T^* f^*)(x^*, z^*, r))}(z^*). \end{aligned}$$

However, even with the images being closed it cannot be concluded that either function has a closed epigraph.

Applying the scalar chain-rule from Theorem 2.3 and Propositions 3.5, 3.6, 4.2, we get the following result.

Theorem 5.3 (Chain-Rule) *Let $g : X \rightarrow \mathcal{P}^\Delta$, $f : Y \rightarrow \mathcal{P}^\Delta$ be two functions and $T : X \rightarrow Y$, $S : Y \rightarrow X$ linear continuous operators.*

(a) *For all $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$ the conjugate of $(g \square \cdot S f)$ is given by the function*

$$\begin{aligned} (g \square \cdot S f)^*(x^*, z^*, r) &= g^*(x^*, z^*, r - \varphi_{f, z^*}^* S^*(x^*)) + f^* S^*(x^*, z^*, \varphi_{f, z^*}^* S^*(x^*)) \\ &\leq \inf_{\substack{r_1 + r_2 = r, \\ r_1, r_2 \in \mathbb{R}}} g^*(x^*, z^*, r_1) + f^* S^*(x^*, z^*, r_2) \end{aligned}$$

and equality holds true if $\text{dom } g \neq \emptyset$ and $\text{dom } f \neq \emptyset$.

(b) *For all $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$ the conjugate of $(g + fT)$ is dominated as follows.*

$$\begin{aligned} (g + fT)^*(x^*, z^*, r) &\leq \inf_{y^* \in Y^*} g^*(x^* - T^* y^*, z^*, r - \varphi_{f, z^*}^*(y^*)) + f^*(y^*, z^*, \varphi_{f, z^*}^*(y^*)) \\ &\leq (g^* \square \cdot T^* f^*)(x^*, z^*, r) \end{aligned}$$

and equality holds true in the second inequality if $\text{dom } g \neq \emptyset$ and $\text{dom } f \neq \emptyset$.

(c) *If $(fT)(x_0) + H(z^*) = Z$ for some $x_0 \in \text{dom } g$ or if both f and g are convex and one of the assumptions in Proposition 5.1 is satisfied for f in an element of $T(\text{dom } g)$, then for all $x^* \in X^*$ there exists an $y^* \in Y^*$ with*

$$\begin{aligned} (g + fT)^*(x^*, z^*, r) &= (g^* \square \cdot T^* f^*)(x^*, z^*, r); \\ &= g^*(x^* - T^* y^*, z^*, r - \varphi_{f, z^*}^*(y^*)) + f^*(y^*, z^*, \varphi_{f, z^*}^*(y^*)). \end{aligned}$$

PROOF.

(a) By Proposition 4.2, the conjugate of a function $h : X \rightarrow \mathcal{P}^\Delta$ can be represented as follows.

$$\forall (x^*, z^*, r) \in X^* \times C^- \times \mathbb{R} : \quad h^*(x^*, z^*) = \{z \in Z \mid \varphi_{h, z^*}^*(x^*) - r \leq -z^*(z)\}.$$

Applying Propositions 3.5, 3.6 and the scalar chain-rule, Theorem 2.3 we may conclude for $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$

$$\begin{aligned} (g \square \cdot S f)^*(x^*, z^*, r) &= H_{(\varphi_{g, z^*}^* + \varphi_{f, z^*}^* S^*)(x^*) - r}(z^*) \\ &= \{z \in Z \mid (\varphi_{g, z^*}^* + \varphi_{f, z^*}^* S^*)(x^*) - r \leq -z^*(z)\}. \end{aligned}$$

If $\text{dom } f = \emptyset$ or $\text{dom } g = \emptyset$, then

$$(\varphi_{g, z^*}^* + \varphi_{f, z^*}^* S^*)(x^*) - r = -\infty,$$

and thus

$$(g \square \cdot S f)^*(x^*, z^*, r) = g^*(x^*, z^*, r \dot{-} \varphi_f^* S^*(x^*)) + f^* S^*(x^*, z^*, \varphi_f^* S^*(x^*)) = Z.$$

Otherwise, both φ_{g,z^*}^* and $\varphi_{f,z^*}^* S^*$ map into the set $\mathbb{R} \cup \{+\infty\}$ and

$$\begin{aligned} & (\varphi_{g,z^*}^* \dot{+} \varphi_{f,z^*}^* S^*)(x^*) - r \\ &= (\varphi_{g,z^*}^*(x^*) \dot{-} (r \dot{-} \varphi_{f,z^*}^* S^*(x^*))) \dot{+} (\varphi_{f,z^*}^* S^*(x^*) \dot{-} \varphi_{f,z^*}^* S^*(x^*)) \end{aligned}$$

and a careful case study of $z^* = 0$ and $\varphi_{f,z^*}^* S^*(x^*) = +\infty$ gives

$$\begin{aligned} & (g \square \cdot S f)^*(x^*, z^*, r) \\ &= H_{\varphi_{g,z^*}^*(x^*) \dot{-} (r \dot{-} \varphi_{f,z^*}^* S^*(x^*))}(z^*) + H_{\varphi_{f,z^*}^* S^*(x^*) \dot{-} \varphi_{f,z^*}^* S^*(x^*)}(z^*) \\ &= g^*(x^*, z^*, \varphi_{f,z^*}^* S^*(x^*)) + f^* S^*(x^*, z^*, \varphi_{f,z^*}^* S^*(x^*)). \end{aligned}$$

The second inclusion is immediate, if $\text{dom } f$ or $\text{dom } g = \emptyset$. In case $\varphi_{g,z^*}^*(x^*) = +\infty$ or $\varphi_{f,z^*}^* S^*(x^*) = +\infty$, then for all $r_2 \in \mathbb{R}$

$$\begin{aligned} & g^*(x^*, z^*, r \dot{-} \varphi_{f,z^*}^* S^*(x^*)) + f^* S^*(x^*, z^*, \varphi_{f,z^*}^* S^*(x^*)) \\ &= g^*(x^*, z^*, r - r_2) + f^* S^*(x^*, z^*, r_2) = \emptyset. \end{aligned}$$

If both $\varphi_{g,z^*}^*(x^*)$ and $\varphi_{f,z^*}^* S^*(x^*) \in \mathbb{R}$, then equality is proven by calculation.

- (b) Applying Propositions 4.2, 3.5, 3.6 and the scalar chain-rule, Theorem 2.3 we may conclude for $(x^*, z^*, r) \in X^* \times C^- \times \mathbb{R}$

$$(g + fT)^*(x^*, z^*, r) \supseteq H_{(\varphi_{g,z^*}^* \square \cdot T^* \varphi_{f,z^*}^*)(x^*) - r}(z^*) \supseteq H_{(\varphi_{g,z^*}^* \square \cdot T^* \varphi_{f,z^*}^*)(x^*) - r}(z^*)$$

and by the same arguments as above

$$\begin{aligned} & H_{(\varphi_{g,z^*}^* \square \cdot T^* \varphi_{f,z^*}^*)(x^*) - r}(z^*) \\ & \supseteq \inf_{y^* \in Y^*} (g^*(x^* - T^* y^*, z^*, r \dot{-} \varphi_{f,z^*}^*(y^*)) + f^*(y^*, z^*, \varphi_{f,z^*}^*(y^*))) \\ & \supseteq (g^* \square \cdot T^* f^*)(x^*, z^*, r). \end{aligned}$$

Equality is shown by a case study with $\varphi_{f,z^*}^*(y^*) = +\infty$ when $\text{dom } f \neq \emptyset$ and $\text{dom } g \neq \emptyset$.

- (c) Applying the scalar chain-rule,

$$(g + fT)^*(x^*, z^*, r) = H_{(\varphi_{g,z^*}^* \square \cdot T^* \varphi_{f,z^*}^*)(x^*) - r}(z^*) = (g^* \square \cdot T^* f^*)(x^*, z^*, r)$$

holds true under the given assumptions and for all $x^* \in X^*$ there is $y^* = T^* x^*$ such that

$$\begin{aligned} & (g + fT)^*(x^*, z^*, r) \\ &= g^*(x^* - T^* y^*, z^*, r \dot{-} \varphi_{f,z^*}^*(y^*)) + f^*(y^*, z^*, \varphi_{f,z^*}^*(y^*)). \end{aligned}$$

□

If $z^* \in C^- \setminus \{0\}$, then

$$(g + fT)^*(x^*, z^*, r) \supseteq g^*(x^* - T^*y^*, z^*, 0) + f^*(y^*, z^*, 0) + H_{-r}(z^*)$$

and under the assumptions of Theorem 5.3 (c),

$$(g + fT)^*(x^*, z^*, r) = g^*(x^* - T^*y^*, z^*, 0) + f^*(y^*, z^*, 0) + H_{-r}(z^*) \neq Z$$

holds true. If additionally $\text{cl}(fT(x_0) + H(z^*)) = Z$, then $(g + fT)^*(x^*, z^*, r) = \emptyset$.

As in the scalar case, equality in Theorem 5.3 (a) and (b) does not hold true with the usual Minkowsky (inf-) addition on the right hand side. Indeed, if $g \equiv \emptyset$ and $fT(x_0) = Z$ for some $x_0 \in X$, then for all $x \in X$ and all $(x^*, z^*, r) \in X^* \times C^- \setminus \{0\} \times \mathbb{R}$ it holds

$$\begin{aligned} (g \square Sf)(x) &= (g + fT)(x) = \emptyset; \\ g^*(x^*, z^*, r) &= Z; \quad f^*S^*(x^*, z^*, r) = T^*f^*(x^*, z^*, r) = \emptyset. \end{aligned}$$

As \emptyset dominates the Minkowsky sum, equality in general is not attained. Notice however, that here as well as in the scalar case (see Theorem 2.3 (d)) we do not assume properness for the strong chain-rule in Theorem 5.3 (c).

Setting $g = 0$ or $X = Y$ and $S = T = \text{id}$, a sum-rule and a multiplication-rule are immediate corollaries of Theorem 5.3.

Corollary 5.4 (Sandwich-Theorem) *Let $T : X \rightarrow Y$ a linear continuous operator and $z^* \in C^-$. Let $g : X \rightarrow \mathcal{P}^\Delta$ and $f : Y \rightarrow \mathcal{P}^\Delta$ be two convex functions such that*

$$\forall x \in X : \quad g(x) \subseteq H(z^*) \dot{-} fT(x)$$

and it exists $x_0 \in \text{dom } g$ such that one of the assumptions in Proposition 5.1 applies for f in $Tx_0 \in Y$. Then there exists $y^ \in Y^*$ and $z_0 \in Z$ such that*

$$\begin{aligned} \forall x \in X : \quad g(x) &\subseteq S_{(T^*y^*, z^*)}(x) + \{-z_0\} \subseteq H(z^*) \dot{-} fT(x); \\ z_0 &\in g^*(T^*y^*, z^*) \cap (H(z^*) \dot{-} f^*(-y^*, z^*)). \end{aligned}$$

If additionally $\text{cl}(g(x_0) + H(z^)) = H(z^*) \dot{-} fT(x_0)$ is fulfilled, then*

$$g^*(T^*y^*, z^*) = \{z_0\} + H(z^*); \quad f^*(-y^*, z^*) = \{-z_0\} + H(z^*).$$

PROOF. By assumption, $\varphi_{g,z^*}, \varphi_{f,z^*} : X \rightarrow \mathbb{R}^\Delta$ are convex and proper, $0 \dot{-} \varphi_{f,z^*}(T(x)) \leq \varphi_{g,z^*}(x)$ for all $x \in X$ and φ_{f,z^*} is continuous in Tx_0 and $x_0 \in \text{dom } \varphi_{g,z^*}$. Thus,

$$\forall x \in X : \quad 0 \leq (\varphi_{g,z^*} \dot{+} \varphi_{f,z^*}T)(x) \tag{5.3}$$

and it exists $y^* \in Y^*$ such that

$$-\infty < s_0 = (\varphi_{g,z^*} + \varphi_{f,z^*}T)^*(0) = \varphi_{g,z^*}^*(T^*y^*) \dot{+} \varphi_{f,z^*}^*(-y^*). \tag{5.4}$$

By (5.3), $s_0 \leq 0$ is valid and by (5.4) it holds

$$\varphi_{g,z^*}^*(T^*y^*) \leq -\varphi_{f,z^*}^*(-y^*) \in \mathbb{R} \tag{5.5}$$

and thus the following holds true for all $x \in X$.

$$\begin{aligned} 0 \dot{-} \varphi_{f,z^*}(Tx) &\leq y^*(Tx) \dot{-} (-\varphi_{f,z^*}^*(-y^*)) \\ &\leq T^*y^*(x) \dot{-} \varphi_{g,z^*}^*(T^*y^*) \leq \varphi_{g,z^*}(x), \end{aligned}$$

and if $\varphi_{g,z^*}(x_0) = 0 \dot{-} \varphi_{f,z^*}(Tx_0)$, then $s_0 = 0$ and equality holds true.

Choose $z_0 \in Z$ such that $-z^*(z_0) = -\varphi_{f,z^*}^*(-y^*) \in \mathbb{R}$. Applying Propositions 3.5, 3.6 and 4.2, we get

$$g^*(T^*y^*, z^*) \supseteq \{z_0\} + H(z^*) = (H(z^*) \dot{-} f^*(-y^*, z^*))$$

and thus

$$g(x) \subseteq S_{(T^*y^*, z^*)}(x) + \{-z_0\} \subseteq H(z^*) \dot{-} fT(x) \quad (5.6)$$

If $\text{cl}(g(x_0) + H(z^*)) = H(z^*) \dot{-} fT(x_0)$ holds, then $\varphi_{g,z^*}(x_0) = \varphi_{f,z^*}(Tx_0)$ is valid, thus

$$g^*(T^*y^*, z^*) = \{z_0\} + H(z^*)$$

and equality holds true in (5.6). \square

Theorem 5.5 (Fenchel-Rockafellar-Duality) *To $g : X \rightarrow \mathcal{P}^\Delta$, $f : Y \rightarrow \mathcal{P}^\Delta$ and a linear continuous operator $T : X \rightarrow Y$ and $z^* \in C^-$, denote*

$$P = \text{cl co } \bigcup_{x \in X} (g(x) + f(Tx)); \quad (5.7)$$

$$D(z^*) = \bigcap_{y^* \in Y^*} H(z^*) \dot{-} (g^*(T^*y^*, z^*) + f^*(-y^*, z^*)). \quad (5.8)$$

(a) *It holds $D(z^*) \supseteq P$.*

(b) *If one of the assumptions in Proposition 5.1 is in force for f in an element in $T(\text{dom } g)$, then $\text{cl}(P + H(z^*)) = D(z^*) \neq \emptyset$ holds and it exists $y_{z^*}^* \in Y^*$ such that*

$$\text{cl}(P + H(z^*)) = H(z^*) \dot{-} (g^*(T^*y_{z^*}^*, z^*) + f^*(-y_{z^*}^*, z^*)) \neq \emptyset.$$

In this case, or if $fT(x_0) = Z$ for some $x_0 \in \text{dom } g$, $P = \bigcap_{z^ \in C^- \setminus \{0\}} D(z^*) \neq \emptyset$ holds true*

and it exists a set $\{y_{z^}^* \in Y^* \mid z^* \in C^- \setminus \{0\}\}$ such that*

$$P = \bigcap_{z^* \in C^- \setminus \{0\}} H(z^*) \dot{-} (g^*(T^*y_{z^*}^*, z^*) + f^*(-y_{z^*}^*, z^*)).$$

PROOF. As $H(0^*) = Z$ and $D(0^*) = Z$, there is nothing left to prove for $z^* = 0$. If $z^* \in C^- \setminus \{0\}$, then the following inequality is met

$$\sup_{y^* \in Y^*} (0 \dot{-} (\varphi_{g,z^*}^*(T^*y^*) \dot{+} \varphi_{f,z^*}^*(-y^*))) \leq \inf_{x \in X} (\varphi_{g,z^*}(x) + \varphi_{f,z^*}(Tx))$$

and equality holds, if φ_{f,z^*} and φ_{g,z^*} are proper functions and φ_{f,z^*} is continuous in $Tx \in Y$ with $x \in \text{dom } \varphi_{g,z^*}$ or if either scalarization attains the value $-\infty$ within the domain of the other. Applying Propositions 3.5, 3.6 and 4.2 proves the statement. \square

We sum up our investigations by stating a set-valued version of the scalar Fundamental Duality Formula as can be found in [33, Theorem 2.7.1].

Theorem 5.6 (Fundamental Duality Formula) *Let $h : X \times Y \rightarrow \mathcal{P}^\Delta$ be convex and such that there exists $x_0 \in X$ with $(x_0, 0) \in \text{dom } h$. Let one of the assumptions in Proposition 5.1 be satisfied for $h(x_0, \cdot) : Y \rightarrow \mathcal{P}^\Delta$ in 0.*

(a) *If h is z^* -proper for $z^* \in C^- \setminus \{0\}$, then*

$$\text{cl} \bigcup_{x \in X} (h(x, 0) + H(z^*)) = \bigcap_{y^* \in Y^*} (H(z^*) - h^*(0, y^*, z^*))$$

and it exists $y_0^ \in Y^*$ such that*

$$\text{cl} \bigcup_{x \in X} (h(x, 0) + H(z^*)) = (H(z^*) - h^*(0, y_0^*, z^*)).$$

Furthermore,

$$\text{cl} (h(\bar{x}, 0) + H(z^*)) = \text{cl} \bigcup_{x \in X} (h(x, 0) + H(z^*)) \quad (5.9)$$

holds for $\bar{x} \in X$ if and only if there is a $\bar{y}^ \in Y^*$ such that*

$$h^*(0, \bar{y}^*, z^*) \supseteq S_{((0, \bar{y}^*), z^*)}(\bar{x}, 0) - h(\bar{x}, 0). \quad (5.10)$$

(b) *If h is z^* -proper for all $z^* \in C^- \setminus \{0\}$, then*

$$\text{cl co} \bigcup_{x \in X} h(x, 0) = \bigcap_{\substack{y^* \in Y^*, \\ z^* \in C^- \setminus \{0\}}} (H(z^*) - h^*(0, y^*, z^*))$$

and it exists a family $\{y_{z^}^* \mid z^* \in C^- \setminus \{0\}\} \subseteq Y^*$ such that*

$$\text{cl co} \bigcup_{x \in X} h(x, 0) = \bigcap_{z^* \in C^- \setminus \{0\}} (H(z^*) - h^*(0, y_{z^*}^*, z^*)).$$

Furthermore,

$$h(\bar{x}, 0) = \text{cl co} \bigcup_{x \in X} h(x, 0) \quad (5.11)$$

holds for $\bar{x} \in X$ if and only if for all $z^ \in C^- \setminus \{0\}$ there is a $\bar{y}_{z^*}^* \in Y^*$ such, that*

$$\forall (x, y) \in X \times Y : \quad h^*(0, \bar{y}_{z^*}^*, z^*) \supseteq S_{((0, \bar{y}_{z^*}^*), z^*)}(\bar{x}, 0) - h(\bar{x}, 0).$$

PROOF.

- (a) If $z^* \in C^- \setminus \{0\}$ and h is z^* -proper or if $z^* = 0$, we can derive that $\varphi_{h,z^*} : X \times Y \rightarrow \mathbb{R}^\Delta$ is convex and proper and $\varphi_{h,z^*}(x_0, \cdot) : Y \rightarrow \mathbb{R}^\Delta$ is continuous in 0, compare Proposition 5.1. Thus in both cases we can apply Theorem 2.2 to $\varphi_{h,z^*} : X \times Y \rightarrow \mathbb{R}^\Delta$ and attain

$$\inf_{x \in X} (\varphi_{h,z^*}(x, 0)) = \sup_{y^* \in Y^*} (0 \dot{-} (\varphi_{h,z^*})^*(0, y^*)),$$

and the existence of $y^* \in Y^*$ such that $\inf_{x \in X} (\varphi_{h,z^*}(x, 0)) = (0 \dot{-} (\varphi_{h,z^*})^*(0, y^*))$. Furthermore,

$$\varphi_{h,z^*}(\bar{x}, 0) = \inf_{x \in X} \varphi_{h,z^*}(x, 0)$$

is satisfied for $\bar{x} \in X$ if and only if $(0, \bar{y}^*) \in \partial \varphi_{h,z^*}(\bar{x}, 0)$, i.e.

$$\exists \bar{y}^* \in Y^* : \forall (x, y) \in X \times Y : (\varphi_{h,z^*})^*(0, \bar{y}^*) \leq (0, \bar{y}^*)(\bar{x}, 0) \dot{-} \varphi_{h,z^*}(\bar{x}, 0).$$

To derive the set-valued result, we apply Propositions 3.5, 3.6 and 4.2 and achieve the desired.

- (b) Assuming the z^* -properness for all $z^* \in C^- \setminus \{0\}$, we obtain above results for all $z^* \in C^- \setminus \{0\}$, thus

$$\text{cl co } \bigcup_{x \in X} h(x, 0) = \bigcap_{z^* \in C^- \setminus \{0\}} \left(\text{cl } \bigcup_{x \in X} h(x, 0) + H(z^*) \right)$$

and it exists $\{y_{z^*}^*\}_{z^* \in C^- \setminus \{0\}} \subseteq Y^*$ such that

$$\text{cl co } \bigcup_{x \in X} (h(x, 0)) = \bigcap_{z^* \in C^- \setminus \{0\}} (H(z^*) \dot{-} h^*(0, y_{z^*}^*, z^*)).$$

Finally, it exists a set $\{y_{z^*}^* \mid z^* \in C^- \setminus \{0\}\}$ such that

$$\forall z^* \in C^- \setminus \{0\} : h^*(0, \bar{y}_{z^*}^*, z^*) \supseteq S_{((0, \bar{y}_{z^*}^*), z^*)}(\bar{x}, 0) \dot{-} h(\bar{x}, 0),$$

if and only if

$$\forall z^* \in C^- \setminus \{0\} : \text{cl } (h(\bar{x}, 0) + H(z^*)) = \text{cl } \bigcup_{x \in X} (h(x, 0) + H(z^*))$$

or equivalently

$$\forall z^* \in C^- \setminus \{0\} : \varphi_{h,z^*}(\bar{x}, 0) = \inf_{x \in X} \varphi_{h,z^*}(x, 0)$$

is satisfied. This is equivalent to

$$\begin{aligned} h(\bar{x}, 0) &= \bigcap_{z^* \in C^- \setminus \{0\}} \text{cl } (h(\bar{x}, 0) + H(z^*)) \\ &= \bigcap_{z^* \in C^- \setminus \{0\}} \text{cl } \bigcup_{x \in X} (h(x, 0) + H(z^*)) \\ &\supseteq \text{cl co } \bigcup_{x \in X} \bigcap_{z^* \in C^- \setminus \{0\}} (h(x, 0) + H(z^*)) \\ &= \text{cl co } \bigcup_{x \in X} h(x, 0) \supseteq h(\bar{x}, 0). \end{aligned}$$

□

The relation in Equation (5.10) is a set-valued variant of the subdifferential formula. In Equation (5.11), $h(\bar{x}, 0)$ can be interpreted as a minimal point of the function $h(\cdot, 0) : X \rightarrow \mathcal{P}^\Delta$, while in Equation (5.9) \bar{x} is minimal with respect to the direction z^* .

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